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HEAT CONDUCTION IN TECHNICAL SYSTEMS

CONSTRUCTION OF A TEMPERATURE FIELD IN THE SECTION OF THE EXTERIOR ANGLE OF ENCLOSING STRUCTURES

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Approximation formulas for determination of the temperature in the exterior angle of enclosing structures and for construction of isotherms in the horizontal section of the angle are proposed; the formulas substitute for the numerical solution of the corresponding heat-conduction problem in a wide range of parameters determining this problem.

In engineering thermal physics, much importance is attached to exterior enclosures of a building with the aim of ensuring conditions comfortable for man. Under severe climatic conditions, they must, primarily, have good thermal-protective properties. In designing enclosing structures, one carries out heat-engineering calculation on determination of the heat loss and the temperature on their interior surfaces to prevent the condensation of steam contained in the indoor air.

A basis for determination of the temperature at the corner of the exterior wall of thickness δ from a homogeneous material with a constant thermal conductivity λ is provided by the differential heat-conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 . \tag{1}$$

We will assume that the coordinates x and y are reckoned from the vertex of the right angle of the exterior wall, as is shown in Fig. 1. The solution of Eq. (1) is found with the following boundary conditions:

(1) on the smooth wall surfaces, we specify the heat flux according to the Newton formula, i.e., the boundary conditions of the IIIrd kind:

$$-\lambda \frac{\partial T}{\partial n} = \alpha \left(T - T_{\mathrm{m}}\right) , \tag{2}$$

where n is the outer normal to the surface and T_{m} is the ambient temperature (indoor and outdoor);

(2) at a fairly large distance (many times larger than δ) from the angle in the cross section of the wall, we specify the symmetry condition, i.e., the boundary condition of the IInd kind:

$$\frac{\partial T}{\partial n} = 0 , \tag{3}$$

here n is the normal to the surface of this cross section.

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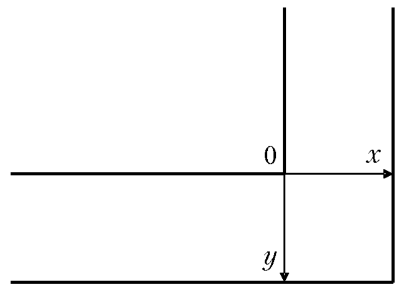

Fig. 1. Exterior angle in the coordinate system x, y.

Let us introduce dimensionless spatial variables $\xi = x/\delta$ and $\eta = y/\delta$ and the dimensionless (reduced) temperature $\Theta = \frac{t_{in} - t}{t_{in} - t_{out}}$, where t_{in} is the indoor temperature and t_{out} is the outdoor temperature. Then the boundary-value problem (1)–(3) will be written as follows;

$$\frac{\partial^2\Theta}{\partial\xi^2} + \frac{\partial^2\Theta}{\partial\eta^2} = 0\ ;$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=0} = \mathrm{Bi}_{in}\Theta\ ,\quad -L \le \eta \le 0\ ;\quad \left.\frac{\partial\Theta}{\partial\eta}\right|_{\eta=0} = \mathrm{Bi}_{in}\Theta\ ,\quad -L \le \xi \le 0\ ;$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = \mathrm{Bi}_{out}(1-\Theta)\ ,\quad -L \le \eta \le 1\ ;\quad \left.\frac{\partial\Theta}{\partial\eta}\right|_{\eta=1} = \mathrm{Bi}_{out}(1-\Theta)\ ,\quad -L \le \xi \le 1\ ; \tag{4}$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=-L} = 0\ ,\quad 0 \le \eta \le 1\ ;\quad \left.\frac{\partial\Theta}{\partial\eta}\right|_{\eta=-L} = 0\ ,\quad 0 \le \xi \le 1\ .$$

Solution of the problem is dependent on the values of Biot similarity numbers: $\mathrm{Bi}_{in} = \frac{\alpha_{in}\delta}{\lambda}$ and $\mathrm{Bi}_{out} = \frac{\alpha_{out}\delta}{\lambda}$. We determine the range of variation in these numbers. In the general case both Biot numbers are more than zero (by definition). Since the angle is a member of the enclosing structure in our case, the internal thermal resistance of the wall must be higher than the thermal resistance of heat transfer from the ambient medium to the wall surface, i.e.,

$$R = \frac{\delta}{\lambda} > R_{in} = \frac{1}{\alpha_{in}} \quad \text{and} \quad R = \frac{\delta}{\lambda} > R_{out} = \frac{1}{\alpha_{out}}\ .$$

It follows that $\mathrm{Bi}_{in} > 1$ and $\mathrm{Bi}_{out} > 1$. Next, from the condition $\alpha_{out} > \alpha_{in}$ it follows that $\mathrm{Bi}_{out} > \mathrm{Bi}_{in}$. Thus, the range of variation in the Biot numbers in our case (heat transfer through the enclosing structure) is

$$\mathrm{Bi}_{out} > \mathrm{Bi}_{in} > 1\ . \tag{5}$$

Numerical solution of the boundary-value problem (4) is found by the establishment method. The method of numerical integration of a parabolic equation with boundary conditions of the general form is used [1]. This method has been selected because of its advantages. In particular, the fourth degree of accuracy of the approximation of the initial heat-conduction equation in the direction of a spatial variable is ensured; it takes smaller computer memory and shorter machine time than methods of low order to obtain the solution with a prescribed exactness. Furthermore, the

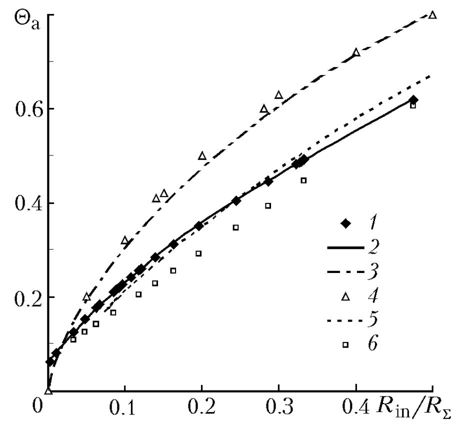

Fig. 2. Relative temperature Θ_a vs. R_{in}/R_Σ:1) numerical calculation; 2) approximation cubic parabola (b); 3–6) data from [2–5] respectively.

function itself, its first derivatives, and the integral of the function act as the sought functions in the method selected (in our case these will be the temperature, the heat-flux density, and the average temperature). This enables us to avoid additional approximation errors in specifying the boundary conditions.

Temperature at the Corner of the Interior Surface. According to the results of solution of the boundary-value problem (4), Fig. 2 plots the relative temperature on the interior surface of the exterior angle $\Theta_a = \dfrac{t_{in} - t_a}{t_{in} - t_{out}}$ versus the ratio R_{in}/R_Σ (curve 1), where $R_\Sigma = R_{in} + R + R_{out}$ is the total thermal resistance of the wall. Passing to Biot numbers, we obtain

$$a = \frac{R_{in}}{R_\Sigma} = \frac{\mathrm{Bi}_{out}}{\mathrm{Bi}_{in} + \mathrm{Bi}_{in}\mathrm{Bi}_{out} + \mathrm{Bi}_{out}},$$

here, $0 < a < 0.5$ throughout the range of variation in the Biot numbers (5). The dependence of Θ_a on the parameter a is approximated, with high accuracy (the correlation factor squared is equal to 0.9998), by the following polynomial:

$$\Theta_a = 2.4778a^3 - 2.7942a^2 + 1.9418a + 0.0619\,, \tag{6}$$

Designers determine this temperature at present either graphically [2–4] or from approximate formulas but in more limited ranges of variation in the parameters determining the problem [3, 5]. Figure 2 gives the approximation cubic parabola (6) (curve 2) and analogous dependences from [2–5] (curves 3–6). It is seen that, according to the results of [2, 3], the reduced temperature Θ_a at the exterior angle is overstated in most of the domain of definition of the parameter a, whereas the corresponding temperature t_a is understated. According to the formula from [5], conversely, the value of the dimensionless temperature Θ_a is lower than that in numerical calculation, whereas t_a is higher. The curve computed from the data of [4] is the closest to the numerical results. Turning our attention to the example from [4], we obtain Θ_a = 0.245 from the approximation (6) for R_{in}/R_Σ = 0.115/(0.115 + 0.903 + 0.043) = 0.108 and the angular temperature t_a = 7.5°C (t_a = 7.6°C in [4]) at t_{in} = 18°C and t_{out} = −25°C. From [2, 3, and 5], we obtain 4.3, 3.5, and 9.7°C respectively.

Thus, for the relative excess temperature Θ_a on the interior surface in the exterior angle, we propose a simple engineering formula (6) which substitutes, with a large degree of accuracy, for the solution of the boundary-value problem (4) in a wide range of the Biot numbers (5) and from which we can also determine the temperature t_a itself at the corner of the enclosing structure from the known indoor and outdoor air temperatures.

Temperature Field in the Section of the Angle. We know of the analytical solution of the heat-conduction equation for the smooth surface of the wall

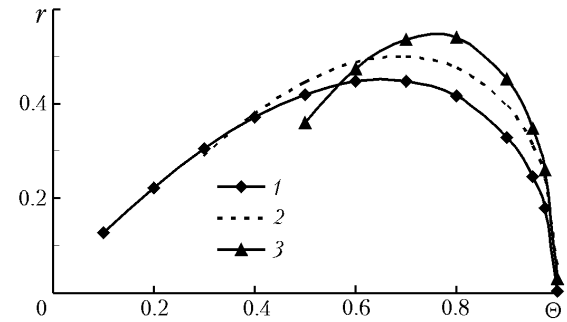

Fig. 3. Dependence of the dimensionless radius of curvature at the vertex of the hyperbolas (9) for different Biot numbers: 1) $\mathrm{Bi_{in}}$ = 400 and $\mathrm{Bi_{out}}$ = 800, 2) 8 and 25, and 3) 2 and 50.

$$\Theta = a + b\xi , \tag{7}$$

where the coefficients a and b are determined from the boundary conditions, from (2), in our case:

$$a = \frac{\mathrm{Bi_{out}}}{\mathrm{Bi_{in}} + \mathrm{Bi_{out}} + \mathrm{Bi_{out}}\mathrm{Bi_{in}}} , \quad b = \mathrm{Bi_{in}} a . \tag{8}$$

On the smooth wall surface, the isotherms are running in parallel to the surfaces bounding the wall.

The solution of the two-dimensional heat-conduction equation (1) approaches the solution (7) asymptotically with distance from the angle; therefore, the asymptotes of the isotherms in the angle's section are the corresponding isotherms for a plane wall:

$$\text{for } \xi \to -\infty \text{ the solution is } \Theta \to a + b\eta \text{ or } \eta \to \frac{\Theta - a}{b} ,$$

$$\text{and for } \eta \to -\infty \text{ the solution is } \Theta \to a + b\xi \text{ or } \xi \to \frac{\Theta - a}{b} .$$

Thus, it may be assumed that the isotherms in the section of the right angle are equilateral hyperbolas whose equations in the coordinate system (ξ, η) will be written in the form

$$\left(\frac{\Theta - a}{b} - \xi\right)\left(\frac{\Theta - a}{b} - \eta\right) = c^2 , \tag{9}$$

where the right-hand side is dependent on the Θ value for a given isotherm, $c = r(\Theta)/\sqrt{2}$, and r is the radius of curvature of the hyperbola (9) at its vertex ($\xi = \eta$).

In particular, for the $\Theta = \Theta_a$ isotherm passing through the interior vertex of the exterior angle ($\xi = \eta = 0$), the radius of curvature is equal to

$$r_a = \frac{\Theta_a - a}{b}\sqrt{2} ,$$

where the reduced angular temperature Θ_a can be calculated from formula (6).

The dependence of the radius of curvature for other isotherms Θ = const ($\Theta_a < \Theta < 1$) represented in the form of (9) along the bisector of the exterior angle ($\xi = \eta$) is rather complex in character according to numerical calculations (Fig. 3). It is proposed that this dependence be approximated by the following formula:

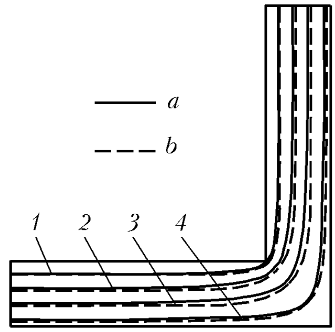

Fig. 4. Isotherms in the section of the exterior angle at $\mathrm{Bi_{in}} = 7$ and $\mathrm{Bi_{out}} = 25$: a) calculation from formula (11); b) numerical solution of problem (4); 1) $\Theta = 0.3$, 2) 0.5, 3) 0.7, and 4) 0.9.

TABLE 1. Maximum Deviation of the Numerical Solution from the Analytical One in Dimensionless Temperature for Different Biot Numbers

$\mathrm{Bi_{out}}$	$\mathrm{Bi_{in}}$								
	2	7	10	20	30	50	100	200	400
800	0.049	0.060	0.061	0.062	0.062	0.062	0.062	0.062	0.062
400	0.049	0.060	0.061	0.061	0.061	0.061	0.062	0.062	
200	0.049	0.060	0.061	0.061	0.061	0.061	0.062		
100	0.048	0.060	0.061	0.061	0.061	0.061			
50	0.048	0.059	0.059	0.060	0.060				
30	0.048	0.059	0.058	0.060					
20	0.047	0.058	0.058						
10	0.044	0.058							
4	0.038								

$$r(\Theta) = \frac{\sqrt{2}}{\mathrm{Bi_{out}}} + \frac{r_{\mathrm{a}} - \dfrac{\sqrt{2}}{\mathrm{Bi_{out}}}}{1 - \Theta_{\mathrm{a}}}(1 - \Theta) + A(1 - \Theta)^B (\Theta - \Theta_{\mathrm{a}})^D ,$$

$$A = 1.2 + \frac{3.4 \cdot \mathrm{Bi_{out}} - 16}{\mathrm{Bi_{out}^2}} \left(\frac{\mathrm{Bi_{out}^2} - 1}{\mathrm{Bi_{in}Bi_{out}} - 1} - 1 \right), \quad B = \frac{a + b}{2}, \quad D = \frac{b}{1 - a} . \tag{10}$$

Thus, the isotherm $\Theta = \mathrm{const}$ ($\Theta_{\mathrm{a}} < \Theta < 1$) in the section of the exterior right angle of a homogeneous wall is described by the following equation (Eq. (9) resolved for η);

$$\eta = \frac{\Theta - a}{b} - \frac{r(\Theta)^2}{2\left(\dfrac{\Theta - a}{b} - \xi\right)} , \tag{11}$$

where $r(\Theta)$ is calculated from formula (10).

Figure 4 gives typical isotherms in the section of the exterior angle: curves a show the result of calculation of the boundary-value problem (4), whereas curves b show the calculation from formula (11).

From formula (11), we can also solve the inverse problem, i.e., compute the temperature Θ from prescribed coordinates (ξ, η) of the point in the region of the exterior-angle section. For this purpose we must solve the transcendental equation (11). To access the exactness of formula (11) we give in Table 1 the maximum deviations in temperature of the numerical solution from the analytical one for different Biot numbers throughout the angle section, i.e., the quantity $\max\limits_{\theta_a \le \Theta < 1} \left| \Theta_{an}(\xi, \eta) - \Theta_{num}(\xi, \eta) \right|$.

The reasons for the differences obtained on comparison of these solutions are as follows. First, the analytical solution has been constructed for the angle with infinite sides, whereas the numerical solution has been obtained for the angle with large sides but, nonetheless, equal to *L*. Second, the approximations (6) and (10) have been used in deriving the analytical formula (11).

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NOTATION

a, *b*, *c*, *A*, *B*, and *D*, approximation parameters; $\mathrm{Bi} = \frac{\alpha\delta}{\lambda}$, Biot similarity number; *L*, length of the angle side, referred to the wall thickness; *n*, outer normal to the boundary surface; *r*, dimensionless radius of curvature of the isotherm; *R*, thermal resistance, $m^2 \cdot K/W$; *t*, temperature, ^oC; *T*, temperature, K; *x*, *y*, spatial coordinates, m; α, coefficient of heat transfer of the wall to the ambient medium, $W/(m^2 \cdot K)$; δ, enclosure-wall thickness, m; λ, thermal conductivity of the material, $W/(m \cdot K)$; $\xi = x/\delta$ and $\eta = \eta/\delta$, dimensionless spatial coordinates; $\Theta = \frac{t_{in} - t}{t_{in} - t_{out}}$, dimensionless (reduced) temperature. Subscripts: m, medium; in, indoor, interior; out, outdoor, exterior; Σ, total; a, angular; an, analytical; num, numerical.

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